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# Formulas for Factorial $N$

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**Abstract.** Burnside's and Stirling's formulas for factorial $N$ are special cases of a family of formulas with corresponding asymptotic series given by E. W. Barnes in 1899. An operational procedure for obtaining these formulas and series is presented which yields both convergent and divergent series and error estimates in the latter case. Two formulas of this family have superior accuracy and the geometric mean is better than either.

**1. Introduction.** Burnside's formula for $N!$ is given by

$$b(N) = (2\pi)^{.5}((N + .5)/e)^{N+.5}. \tag{1}$$

(1) has a number of advantages relative to the usual Stirling formula

$$s(N) = (2\pi e)^{.5}(N/e)^{N+.5}. \tag{2}$$

Thus if $Eb(N)$ and $Es(N)$ are defined by the equations

$$b(N)/N! = 1 + Eb(N): s(N)/N! = 1 - Es(N), \tag{3}$$

then for $N = 1, 2, \ldots,$ both $Eb(N)$ and $Es(N)$ are positive and, practically, $Es(N) = 2Eb(N)$. One can take logarithms of $b(N)$ for $N = 0, 1, \ldots,$ and, indeed, $Eb(0) = .07$.

The computational value of these formulas is based on the associated asymptotic series. The asymptotic series for both $b(N)$ and $s(N)$ are special cases of a family of asymptotic series for $N!$. The classical textbook procedure for obtaining $s(N)$ is based on Euler-Maclaurin summation. For example, in [5] one has a development explicitly based on the properties of the Bernoulli functions. An alternate procedure is presented here, using "operational" methods which produce both convergent and divergent series and error estimates in the latter case. The family of asymptotic series is known. Thus Eq. (28) of this paper is related to Eq. 12, p. 48 of [2], by an obvious change of independent variables and an explicit formula for the remainder term. Equation 12 of [2] is ascribed to E. W. Barnes. Burnside [1], showed that $\log b(N)$ is the initial term of a convergent series for $\log(N!)$ and Wilton [6], generalized Burnside's result to nonintegral values.

The formulas $s(N)$ and $b(N)$ avoid certain difficulties which are associated with the iterative computation of $N!$, when their accuracy is adequate. There are, however, two formulas, corresponding to members of the above-mentioned family of series, which have superior accuracy, and the geometrical mean is even better. In each of these five formulas only one logarithm and antilogarithm is computed.

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**2. Convergent Series for Factorial** $N$. The Burnside series can be derived as follows. Let

$$f(x) = x \log |x| - x. \tag{4}$$

For positive integral values of $k$, consider $f(k + h)$ as a Taylor series at $k$ for $h = .5$ and $h = -.5$. Taking the difference yields

$$f(k + .5) - f(k - .5) = \log k - \sum_{j=1}^{\infty} 1/2j(2j + 1)(2k)^{2j}. \tag{5}$$

Define $\zeta$ by the equation

$$\sum_{k=N+1}^{\infty} 1/k^{2j} = \sum_{k=0}^{\infty} 1/(k + N + 1)^{2j} = \zeta(2j, N + 1). \tag{6}$$

To obtain an expression for $\log N!$, (5) can be summed from 2 to $N$.

$$f(N + .5) - f(1.5) = \log N! - \sum_{j=1}^{\infty} (\zeta(2j, 2) - \zeta(2j, N + 1))/2j(2j + 1)2^{2j}. \tag{7}$$

Combining the terms independent of $N$ into a constant $c$ yields

$$\log N! = f(N + .5) + c - \sum_{j=1}^{\infty} \zeta(2j, N + 1)/2j(2j + 1)2^{2j}. \tag{8}$$

The values of the $\zeta$ function which appear can be estimated by the usual integral test. Thus the convergence and characteristics of the series in the last expression in (8) are readily obtained. Since this expression approaches zero as $N \to \infty$, a comparison with Stirling's formula shows that $c = .5 \log(2\pi)$. With this value of $c$, (8) becomes the Burnside series.

The $\zeta$ functions in (8) appear quite formidable in regard to computation, although they really are not. Cf. [2, Section 1.10, p. 24]. Also the Stirling asymptotic series for $\log N!$ has terms consisting of negative powers of $N$ with relatively small rational coefficients. It would clearly be desirable to have a similar series with negative powers of $N + .5$.

Consider then the difference equation

$$b(x + .5) - b(x - .5) = \log x. \tag{9}$$

A solution, $b(x)$, of this equation, defined for $x \geqslant 1.5$ will yield an expression for $\log N!$

$$\log N! = b(N + .5) + c \tag{10}$$

by summing (9) for $x = 2, \ldots, N$.

We obtain a solution of (9) by "operational methods." We proceed formally and return to a justification later. In terms of the differential operator, $D$, (9) can be written

$$(\exp(.5D) - \exp(-.5D))b(x) = \log x. \tag{11}$$

Integration is equivalent to dividing by $D$, i.e.,

$$(2 \sinh(.5D)/D)b(x) = f(x) + A, \tag{12}$$

where $A$ is a constant of integration. The operator on the left, when applied to a constant, yields the constant. Thus the constant $A$ can be incorporated into $b(x)$. Then

$$b(x) = (.5D/\sinh(.5D))f(x). \tag{13}$$

Let $z$ denote a complex variable. The expansion of the meromorphic function $\csc z$ in terms of its poles yields (see [3, p. 463])

$$z/\sin z = 1 + \sum_{n=1}^{\infty} (-1)^n 2z^2/(z^2 - n^2\pi^2). \tag{14}$$

Substituting $z = .5ix$ yields

$$.5x/\sinh .5x = 1 + \sum_{n=1}^{\infty} (-1)^n 2x^2/(x^2 + 4n^2\pi^2). \tag{15}$$

For $x = D$, using (15) in (13) yields

$$b(x) = f(x) + \sum_{n=1}^{\infty} (-1)^n y_n(x), \tag{16}$$

where

$$(D^2 + 4n^2\pi^2)y_n(x) = 2D^2 f(x) = 2/x. \tag{17}$$

For positive $x$, variation of parameters yields a solution of (17) which goes to zero as $x$ goes to infinity, i.e.,

$$y_n(x) = (1/\pi n)\int_x^{\infty} (1/t)\sin(2\pi n(t - x))\, dt. \tag{18}$$

By changing the variable of integration this can also be expressed as

$$y_n(x) = (1/\pi n)\int_0^{\infty} (\sin u/(u + 2\pi n x))\, du. \tag{19}$$

Since any two distinct solutions of (17) must differ by a harmonic oscillation of nonzero amplitude, (19) is the only solution of (17) which goes to zero as $x$ goes to plus infinity.

Now one can readily show that for $\sigma \geqslant 0$

$$\begin{aligned} &\int_0^{\infty} (\sin u/(u + \sigma))\, du \\ &\quad = \int_0^{\pi} \sin u \left( \sum_{k=0}^{\infty} \pi/(2k\pi + u + \sigma)((2k + 1)\pi + u + \sigma) \right) du, \end{aligned} \tag{20}$$

and consequently, for $x \geqslant 0$, $y_n(x) \geqslant 0$ and for each $x$, $y_n(x)$ decreases as $n$ increases, and, for each $n$, $y_n(x)$ is a monotonically decreasing function of $x$ for $x \geqslant 0$. Thus the summation in (16) is for each $x$ an alternating series of decreasing terms, and, indeed, one has uniform convergence for $x \geqslant 0$. Since the summation in (16) approaches zero as $x \to \infty$, comparison with Stirling's formula shows that the constant $c$ in (10) has value $.5 \log 2\pi$.

**3. The Asymptotic Series.** The series (16) for $b(x)$ with terms given by (19) does not satisfy our requirement for a series in negative integral powers of $N + .5$. We proceed to obtain an asymptotic series of this type with a remainder estimate.

Repeated integration by parts yields

$$y_n(x) = (1/n\pi) \sum_{j=1}^{k} (-1)^{j+1}(2j-2)!/(2\pi nx)^{2j-1} \tag{21}$$
$$+ (-1)^k(2k)!\,(1/n\pi)\int_0^\infty \left(\sin u/(2\pi nx+u)^{2k+1}\right) du$$

and

$$\sum_{n=1}^{\infty} (-1)^n y_n(x) = \sum_{j=1}^{k} a_j(2j-2)!/x^{2j-1} + R(k,x), \tag{22}$$

where

$$R(k,x) = (-1)^k(2k)! \sum_{n=1}^{\infty} (-1)^n(1/n\pi)\int_0^\infty \left(\sin u/\,(2\pi nx+u)^{2k+1}\right) du \tag{23}$$

and

$$a_j = 2(-1)^{j+1} \sum_{n=1}^{\infty} (-1)^n/(2\pi n)^{2j}. \tag{24}$$

But $a_j$ can also be evaluated as follows. The function

$$.5x/\sinh .5x = 1/\left(1 + (.5x)^2/3! + (.5x)^4/5! + \cdots\right) \tag{25}$$

is analytic at $x = 0$ and has a Taylor expansion

$$1 + \sum_{j=1}^{\infty} c_j x^{2j} \tag{26}$$

valid for $|x| < 2\pi$, with rational $c_j s$ which are readily evaluated. But the left-hand side of (25) is also given in (15). Now, if we factor out $4n^2\pi^2$ in the denominator of the terms in (15), we can express the function as a power series in $x/2\pi n$ and obtain ultimately

$$.5x/\sinh .5x = 1 + 2\sum_{j=1}^{\infty} (-1)^{j+1}\left(\sum_{n=1}^{\infty} (-1)^n/\,(2\pi n)^{2j}\right) x^{2j} \tag{27}$$

for $|x| < 2\pi$. Comparing with (26) shows that $a_j = c_j$. However, we also have $D^{2j}f(x) = (2j-2)!/x^{2j-1}$, so that (16) and (22) yield

$$b(x) = f(x) + \sum_{j=1}^{k} c_j D^{2j}f(x) + R(k,x). \tag{28}$$

If, in (28), we ignore the remainder term and let $k = \infty$, we obtain an expression which corresponds to using (26) as the function of $D$ in (13). This is, of course, a most naive way to solve (11) and yields a divergent series. On the other hand, the properties of $R(k, x)$ yield very useful results since the $c_j$ can be readily calculated.

The argument, in the paragraph containing (20), is based simply on the fact that $\sin(u + \pi) = -\sin u$ and readily generalizes to yield properties of the summation in (23) in place of the summation in (16). In particular, the summation in (23) is always negative, and for $k$ fixed this summation approaches zero as $x \to \infty$. Hence the $f(x)$ terms in (28) yield an asymptotic expression for $b(x)$, and $R(k, x)$ has the sign

$(-1)^{k+1}$. Hence, if one adds a term to the asymptotic expression, the remainder changes sign, i.e., each term overshoots and must be larger than the previous remainder. These are, of course, computationally desirable properties of the asymptotic expression.

**4. On Justifying the Formal Procedure.** We now return to justifying the formal procedure of Section 2. This can be done by using the Fourier transform to express the operator $D$ on generalized functions, that is, using the methods described in [4]. However, to justify our procedure it is only necessary to show that (9) holds for $x = 2, 3, \ldots$ for $b$ given by (16) and (19). We now show that (9) holds for $x > .5$.

By making the change of variable $u = 2\pi n v$ and manipulating the limits of integration, we obtain

$$(29)\quad y_n(x + .5) - y_n(x - .5) = \left((-1)^{n+1}/n\pi\right)\int_{-.5}^{.5} (\sin 2\pi n u/\,(u + x))\, du.$$

To make the required summation in (16), we consider

$$(30)\quad \begin{aligned} h(u) &= -\sum_{n=1}^{\infty} \sin 2\pi n u/n = \operatorname{Im}\log(1 - \exp(2\pi i u)) \\ &= \operatorname{Im}(\log(2\sin \pi u) + \log(\sin \pi u - i\cos \pi u)) \\ &= .5(1 - \operatorname{sign} u)\pi + \pi(u - \tfrac{1}{2}) \end{aligned}$$

using half angle formulas. Using this summation formula one obtains

$$(31)\quad \begin{aligned} &\sum_{n=1}^{\infty} (-1)^n (y_n(x + .5) - y_n(x - .5)) \\ &\qquad = 1 + \log x + (x - .5)\log(x - .5) - (x + .5)\log(x + .5), \end{aligned}$$

which implies (9).

The Fourier transform approach requires that $y_n(x)$ be defined for negative $x$. If one considers integration across a simple pole, as given by the Cauchy limit, then (19) yields that for $x < 0$

$$(32)\quad y_n(x) = \cos(2\pi n x)/n - y_n(x).$$

The summation argument of (30) then yields for $x < 0$

$$(33)\quad b(x) = -\log(2\,|\cos \pi x\,|) - b(|\,x\,|).$$

**5. The Family of Asymptotic Series.** The equations (10), (28) and the evaluation of the constant $c$ at the end of Section 2 yield the asymptotic series for $\log N!$

$$(34)\quad \begin{aligned} \log N! = {} & f(N + .5) + .5\log(2\pi) - 1/24(N + .5) \\ & + 7/K_3(N + .5)^3 - 31/K_5(N + .5)^5 + 127/K_7(N + .5)^7 - \cdots, \end{aligned}$$

where $K_3 = 2^6 \times 3^2 \times 5 = 2880$, $K_5 = 2^7 \times 3^2 \times 5 \times 7 = 40320$, and $K_7 = 2^{11} \times 3 \times 5 \times 7 = 215040$.

The argument used above will also yield other asymptotic series. For example, if we replace (9) by

$$(35)\quad s(x) - s(x - 1) = \log x,$$

the equivalent of (25) is

$$x/(1-\exp(-x)) = 1/(1 - x/2! + x^2/3! - \cdots), \tag{36}$$

and the equivalent of (34) is

$$\log N! = f(N) + .5\log N + .5\log 2\pi + 1/12N - 1/360N^2 + \cdots, \tag{37}$$

which is the usual Stirling series for $\log N!$.

In general, we can replace (9) by

$$g(x + .5 + \alpha) - g(x - .5 + \alpha) = \log x. \tag{38}$$

One expands the function $\exp(-\alpha x)/2\sinh .5x - 1/x + \alpha$ in terms of its poles, and one obtains, corresponding to (15) in the previous argument,

$$\begin{aligned} &x\exp(-\alpha x)/2\sinh .5x \\ &\quad = 1 - \alpha x + 2\sum_{n=1}^{\infty}(-1)^n x^2(\cos 2\pi n\alpha - x\sin 2\pi n\alpha/2\pi n)/\left(x^2 + (2\pi n)^2\right). \end{aligned} \tag{39}$$

To obtain the equivalent to (16), we use both $y_n$, defined by (19), and

$$z_n = -Dy_n/2\pi n = (1/\pi n)\int_0^{\infty}(\cos u/(u + 2\pi nx))\,du. \tag{40}$$

The equivalent of (16) is then

$$g(x) = f(x) - \alpha\log x + \sum_{n=1}^{\infty}(-1)^n\left(y_n\cos(2\pi n\alpha) + z_n\sin(2\pi n\alpha)\right). \tag{41}$$

A rather obvious modification of the argument associated with Eqs. (29) and (30) shows that (41) satisfies (38).

The Taylor expansion corresponding to (26) is

$$\begin{aligned} &x\exp(-\alpha x)/2\sinh .5x \\ &\quad = 1 - \alpha x + \tfrac{1}{2}\left(\alpha^2 - \tfrac{1}{12}\right)x^2 + (\alpha/6)(.25 - \alpha^2)x^3 \\ &\qquad + (1/24)\left((\alpha^2 - .25)^2 - 1/30\right)x^4 \\ &\qquad - (\alpha/120)\left((\alpha^2 - 5/12)^2 - 1/36\right)x^5 + \cdots, \end{aligned} \tag{42}$$

which yields the asymptotic series for $\log N!$

$$\begin{aligned} \log N! &= f(N + .5 + \alpha) - \alpha\log(N + .5 + \alpha) + .5\log 2\pi \\ &\quad + .5\left(\alpha^2 - 1/12\right)/(N + .5 + \alpha) - \alpha(.25 - \alpha^2)/6(N + .5 + \alpha)^2 \\ &\quad + \left((\alpha^2 - .25)^2 - 1/30\right)/12(N + .5 + \alpha)^3 \\ &\quad + \alpha\left((\alpha^2 - 5/12)^2 - 1/36\right)/20(N + .5 + \alpha)^4 + \cdots. \end{aligned} \tag{43}$$

The asymptotic character and the remainder of (43) is readily obtained by the methods used after (28) above, but the effect of the remainder is more complicated and depends on $\alpha$.

**6. The Formulas.** Define for $-.5 \leqslant \alpha < .5$

$$M(N,\alpha) = (N + .5)\log(N + .5 + \alpha) - (N + .5 + \alpha) + .5\log 2\pi, \tag{44}$$

so that $\log N! = M(N,\alpha) + o(1)$ as $N \to \alpha$. For fixed $N$, $M$ has a maximum at $\alpha = 0$ and a minimum at the lower extreme point. If $M(N,\alpha)$ is used as an approximation for log $N!$, then by (43) the error is $O(1/N)$ for large $N$ except for $\alpha = \pm(1/12)^{1/2}$, where the error is essentially $-\alpha/36(N + .5 + \alpha)^2$.

We define the $\alpha$ formula for $N!$ as $b_\alpha(N) = \exp(M(N, \alpha))$. We have $b_0(N) = b(N)$, $b_{-.5}(N) = s(N)$. Referring to Eq. (3), we have for large $N$

(45) $$Eb(N) \sim 1/24(N + .5); \quad Es(N) \sim 1/12N.$$

In general

(46) $$b_\alpha(N) = (2\pi)^{.5} e^{-\alpha}((N + .5 + \alpha)/e)^{N+.5}.$$

Let $d = (1/12)^{.5}$. If $Ed(N)$ is defined by

(47) $$b_\alpha(N)/N! = 1 + Ed(N),$$

then for large $N$, $Ed(N) \sim d/36(N + .5 + d)^2$. For $\alpha = -d$ the error has the opposite sign and is larger.

The geometric mean of $b_\alpha$ and $b_{-\alpha}$ is given by

(48) $$gm(N) = (b_d b_{-d})^{.5} = (2\pi)^{.5}\left(((N + .5)^2 - 1/12)/e^2\right)^{.5N+.25}.$$

If $gm(N)/N! = 1 - Eg(N)$, then for large $N$

(49) $$Eg(N) \sim 1/240(N + .5)^3.$$

*Table of Fractional Errors*

| N | N! | Es | Eb | Ed | Eg |
|---|---|---|---|---|---|
| 0 | 1 | 1.0 | .0750476 | .0116301 | .0285848 |
| 1 | 1 | .077863 | .027508 | .0024793 | .0011684 |
| 2 | 2 | .040498 | .016655 | .0010333 | .00026056 |
| 3 | 6 | .027298 | .01192 | .00056145 | 9.5981E¯5 |
| 4 | 24 | .020576 | .0092757 | .00035165 | 4.5374E¯5 |
| 5 | 120 | .016507 | .00759 | .00024064 | 2.4914E¯5 |
| 6 | 720 | .01378 | .006422 | .00017492 | 1.5115E¯5 |
| 7 | 5040 | .011826 | .0055653 | .00013284 | 9.8486E¯6 |
| 8 | 40320 | .010357 | .00491 | .0001043 | 6.7697E¯6 |
| 9 | 3.6288E5 | .0092128 | .0043928 | 8.4052E¯5 | 4.8512E¯6 |
| 10 | 3.6288E6 | .008296 | .003974 | 6.9174E¯5 | 3.5941E¯6 |
| 15 | 1.3077E12 | .0055393 | .0026911 | 3.2266E¯5 | 1.1182E¯6 |
| 20 | 2.4329E18 | .0041577 | .0020343 | 1.86E¯5 | 4.8339E¯7 |
| 25 | 1.5511E25 | .0033276 | .0016352 | 1.2082E¯5 | 2.5101E¯7 |
| 30 | 2.6525E32 | .0027738 | .001367 | 8.4737E¯6 | 1.4678E¯7 |
| 35 | 1.0333E40 | .0023781 | .0011743 | 6.2696E¯6 | 9.3471E¯8 |
| 40 | 8.1592E47 | .0020811 | .0010293 | 4.8262E¯6 | 6.2658E¯8 |
| 45 | 1.1962E56 | .0018501 | .00091614 | 3.8289E¯6 | 4.4519E¯8 |
| 50 | 3.0414E64 | .0016653 | .0008254 | 3.1121E¯6 | 3.2295E¯8 |

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